Steinberg Modules & Arithmetic Groups

Examples : $SL_n\mathbb{Z}$, $\Gamma_n(p) = \ker(SL_n\mathbb{Z} \xrightarrow{\text{mod } p} SL_n\mathbb{F}_p)$

$SL_n R$ $\quad$ R: no. ring (eg: $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$)

$Sp_{2n} R$

$R = \mathbb{Z}$	$\operatorname{rk} H^{\binom{n}{2}}(\Gamma_n(3)) = 3^{\binom{n}{2}}$ (Lee - Szczarba)	$\operatorname{rk} H^{\binom{n}{2}}(\Gamma_n(5))$: recursive in n... $> 2^{n-1} 5^{\binom{n}{2}}$ (Miller - Patzt - Putman)
$R = \mathbb{Z}[i]$	$\operatorname{rk} H^{n^2-n}(\Gamma_n(1+2i)) = 5^{\binom{n}{2}}$	[P.] $\operatorname{rk} H^{n^2-n}(\Gamma_n(3))$: recursive in n... $> 2^{n-1} 9^{\binom{n}{2}}$
$R = \mathbb{Z}[\omega]$	$\operatorname{rk} H^{n^2-n}(\Gamma_n(1+3\omega)) = 7^{\binom{n}{2}}$	[P.] $\operatorname{rk} H^{n^2-n}(\Gamma_n(1+4\omega))$: recursive in n... $> 2^{n-1} (13)^{\binom{n}{2}}$

Q : What's special about these H^* degrees?
What is governing these calculations?
(Teaser : Depends on units of $R/(p)$ vs R ...)

Borel - Serre Duality

R : no. ring, $\quad \Gamma <_{\text{fin ind.}} SL_n R$ torsion-free

$$H^{\nu - i}(\Gamma) \cong H_i(\Gamma; \mathcal{D})$$

dualising module

$\nu = \nu(R)$: quadratic in n] "top degree"

$R = \mathbb{Z}$: $\nu = \binom{n}{2}$

$R = \mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$: $\nu = n^2 - n$

This talk: R Euclidean domain, $\Gamma = \Gamma_n(p)$ $p \in R$ prime

$$H^{\nu}(\Gamma_n(p)) \cong H_0(\Gamma_n(p); \mathcal{D}) \cong (\mathcal{D})_{\Gamma_n(p)}$$

$\mathcal{D}$: The Steinberg Module

Steinberg Modules

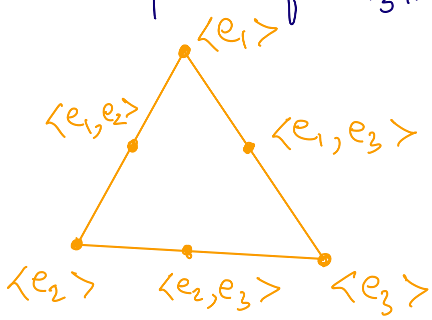
representation of spl. linear group

$\mathrm{St}_n \mathbb{F}$ $\mathbb{F}$: field

$$\mathrm{St}_n \mathbb{F} = \tilde{H}_{n-2}(T_n \mathbb{F})$$

Tits building

$n=3$: Subcplex of $T_3 \mathbb{F}$



apartment

$T_3 \mathbb{F}$ built by "gluing such spheres together"

Vertices of $T_n \mathbb{F}$: $0 \subsetneq V \subsetneq \mathbb{F}^n$

k-simplices : $0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{F}^n$

Thm [Solomon – Tits] : $T_n \mathbb{F} \simeq \bigvee S^{n-2}$

$\mathrm{St}_n \mathbb{F} \cong \tilde{H}_{n-2}(T_n \mathbb{F})$ generated by apt. classes $[\underbrace{v_1, v_2, \ldots, v_n}_{\text{basis of } \mathbb{F}^n}]$

This talk : R Euclidean domain, $\Gamma = \Gamma_n(p)$ $p \in R$ prime

$$H^{\nu}(\Gamma_n(p)) \overset{\text{Borel-Serre}}{\cong} H_0(\Gamma_n(p); \mathrm{St}_n K) \quad K = \mathrm{Frac}(R)$$

$$\cong (\mathrm{St}_n K)_{\Gamma_n(p)}$$

Q: How do we get bounds on $(\mathrm{St}_n K)_{\Gamma_n(p)}$?

Fix $R = \mathbb{Z}$, $p \in \mathbb{Z}$

$$T_n \mathbb{Q} \longrightarrow T_n \mathbb{F}_p$$

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{Q}^n$$

$$\updownarrow$$

$$0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_k \subsetneq \mathbb{Z}^n \quad \overset{\text{mod } p}{\longmapsto} \quad \text{flag in } \mathbb{F}_p$$

direct summands

$$\boxed{\rightsquigarrow \text{Get } (\mathrm{St}_n \mathbb{Q})_{\Gamma_n(p)} \longrightarrow \mathrm{St}_n \mathbb{F}_p : \mathrm{rk}\ p^{\binom{n}{2}}}$$

Lee-Szczarba : inj. when $p = 3$.

Key : units of $\mathbb{F}_3$ lift to units of $\mathbb{Z}$.

Miller-Patzt-Putman : false for $p \geq 5$.

$\boxed{Q}$ What does $T_n\mathbb{Q} \to T_n\mathbb{F}_p$ forget?

Eg: $n=2$, $p=5$

$$\underbrace{\left\langle \begin{bmatrix}1\\0\end{bmatrix} \right\rangle, \left\langle \begin{bmatrix}2\\5\end{bmatrix} \right\rangle}_{T_2\mathbb{Q}} \longmapsto \underbrace{\left\langle \begin{bmatrix}1\\0\end{bmatrix} \right\rangle = \left\langle \begin{bmatrix}2\\0\end{bmatrix} \right\rangle}_{T_2\mathbb{F}_5}$$

But $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}$ lie in different $\Gamma_2(5)$-orbits.

Want version of $T_2\mathbb{F}_5$ that distinguishes b/w $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}2\\0\end{bmatrix}$

Idea: Identify line generators upto $(\pm)$-sign

New complex: $T_n^{\pm}\mathbb{F}_p$ Vertices: (V, ω)

Rmk: $T_n^{\pm}\mathbb{F}_3 = T_n\mathbb{F}_3$ $\omega \in \Lambda^{top} V$, upto $\pm$ sign

$$(T_n\mathbb{Q})_{\Gamma_n(p)} \to T_n^{\pm}\mathbb{F}_p$$

Conjecture (Lee-Szczarba): $(St_n\mathbb{Q})_{\Gamma_n(p)} \to St_n^{\pm}\mathbb{F}_p$ is an iso

Thm [Miller-Patzt-Putman '21]:

- Always surjective
- Iso for $p = 2, 3, \boxed{5}$.
- not injective for $p \geq 7$.
- Recursive formula for $St_n^{\pm}\mathbb{F}_p$

$$\boxed{\text{Thus } H^{\binom{n}{2}}(\Gamma_n(5)) \cong St_n^{\pm}\mathbb{F}_p}$$

<u>Proof idea</u> : • Generating set for $St_n^{\pm}\mathbb{F}_p \rightsquigarrow$ surjectivity

• Presentation for $St_n^{\pm}\mathbb{F}_p$, $p \leq 5 \rightsquigarrow$ injectivity using high connectivity of simplicial complexes $BDA_n^{\pm}\mathbb{F}_p$

" $\mathbb{F}_5$ does not have too many more units than $\mathbb{Z}$ "

<u>Thm [P.]</u> : Adapt MPP's techniques to compute

$$H^{n^2-n}(\Gamma_n(p)) \text{ for :}$$

$$p = 3 \in \mathbb{Z}[i]$$

$$p = 4\omega+1,\ 4\omega+3 \in \mathbb{Z}[\omega]$$

These are special instances of a more general result :

<u>Thm [P.]</u> R Euclidean no. ring, $p \in R$ prime.

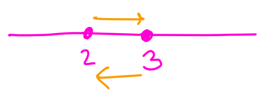
$$\mathbb{F} = R/(p), \quad U = \mathrm{im}(R^\times \longrightarrow \mathbb{F}^\times)$$

$$H^{\nu}(\Gamma_n(p)) \longrightarrow St_n^U\mathbb{F} \quad \text{is surj.}$$

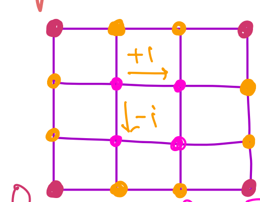
Inj. if $U = \mathbb{F}^\times$, or :
($eg.$ $p = 3, R = \mathbb{Z}$)

① $\mathbb{F}^\times / U \cong \mathbb{Z}/2$ ($eg.$ $p=5, R=\mathbb{Z}$: $(\pm 2)(\pm 2) \equiv \pm 1 \mod 5$)

② $\mathbb{F}^\times \setminus U$ "additively connected by U"



$p = 5,\ R = \mathbb{Z}$



$p = 3,\ R = \mathbb{Z}[i]$

allows us to say different choices of a certain map are homotopic

③ $\mathbb{F}$ additively generated by U allows for an induction argument

④ $2 \in \mathbb{F}^{\times} \setminus U$ or ______________

⑤ $BDA_2^U \mathbb{F}$ is 1-connected

Closing Remarks

- $Sp_{2n} R$ has a similar duality story, symplectic Tits building
- Combinatorics even more complicated than for SL_n ; same techniques not feasible
- Using different ideas : $p = 3$ analogue for $Sp_{2n} R$

Eg:

$R = \mathbb{Z}$	$\mathrm{rk}\ H^{n^2}(\Gamma_n(3)) = 3^{n^2}$
$R = \mathbb{Z}[i]$	$\mathrm{rk}\ H^{2n^2-n}(\Gamma_n(1+2i)) = 5^{n^2}$
$R = \mathbb{Z}[\omega]$	$\mathrm{rk}\ H^{2n^2-n}(\Gamma_n(1+3\omega)) = 7^{n^2}$